Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics

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Lecture 3

Reminder: C^*-alg. : $\|a^*a\| = \|a\|^2$, $\|ab\| \leq \|a\|\,\|b\|$ - only Banach

$R_a(z) := (a - z.1)^{-1}$: $\exists \underset{\text{def}}{\Leftrightarrow} z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$

$\sigma_{\mathcal{A}}(a)$ - compact, $\neq \emptyset$, independent from $\mathcal{A}$.

$\rho_{\mathcal{A}}(a) = \sup\{|z| \mid z \in \sigma_{\mathcal{A}}(a)\}$, $\rho_{\mathcal{A}}(a) = \|a\|$ for $a^* = a$

Positivity (положителност) in C^*-algebras

Cone in a complex vector space V : $K \subseteq V$ s.t. $v \in K$, $\lambda > 0 \Rightarrow \lambda v \in K$

Positive element in a $*$-algebra $\mathcal{A}$ is called any linear combination

$$\lambda_1 a_1^* a_1 + \cdots + \lambda_n a_n^* a_n \, , \quad \lambda_1, \ldots, \lambda_n > 0$$

Notation: $c \geq 0$ if c is positive.

Clearly: the positive elements form a cone.

An (algebraic) linear functional $\omega : \mathcal{A} \to \mathbb{C}$ on a $*$-algebra is called positive iff

$$\omega(a^*a) \geq 0 \text{ for all } a \in \mathcal{A}$$

Clearly, ω is positive $\Leftrightarrow$ $\omega(c) \geq 0 \quad \forall c \geq 0$.

Remark The posit. lin. functs. form a cone that is an example of the notion of a dual cone;

for $S \subseteq V$ - subset, $S^\circ := \{\omega : V \xrightarrow[\text{lin.}]{} \mathbb{C} \mid \omega(v) \geq 0 \ \forall v \in V\}$.

If V - finite dimensional then $(S^\circ)^\circ$ = convex cone envelope of S
изпъкнала конусна обвивка

Def. A lin. funct. $\omega : \mathcal{A} \to \mathbb{C}$ on a $*$-alg. $\mathcal{A}$ is called real iff $\omega(a^*) = \overline{\omega(a)}$.
Equivalently, $a = a^* \Rightarrow \omega(a) \in \mathbb{R}$.

Remark: If in a $*$-alg. $\mathcal{A}$ $\forall$ Hermitian a is a real linear combination of positive elements then $\forall$ positive lin. funct. is real.

Later we shall prove: in a C^*-alg. $\forall$ Hermit. a : $a = b - c$; $b, c \geq 0$

<u>Theorem 3.1</u> Let $\mathcal{A}$ be a (unital) C^*-algebra. If ω is a positive lin. funct. (real) on $\mathcal{A}$ then φ is bounded (and $\|\omega\| = \omega(1)$).

<u>Def.</u> A sesquilinear form (полу-линейна форма) in a compl. vect. space V:

$$K : \underset{\text{antilinear}}{V} \times \underset{\text{linear}}{V} \to \mathbb{C} \quad K(v, \alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 K(v, u_1) + \alpha_2 K(v, u_2)$$
$$K(\alpha_1 v_1 + \alpha_2 v_2, u) = \alpha_1 K(v_1, u) + \alpha_2 K(v_2, u)$$

Hermitian form (ермитова форма) - if in addition: $K(u,v) = \overline{K(v,u)}$

K is called positive semidefinite iff: $K(u,u) \geq 0 \quad \forall u \in V$.

<u>Reminder:</u> (Cauchy - Schwarz inequality / Неравенсово на Коши - Шварц)
for a positive Hermitian K:
$$\forall u, v \in V \quad |K(u,v)|^2 \leq K(u,u)\, K(v,v)$$

<u>Corollary 3.2.</u> For $\forall$ real positive lin. funct. ω on $\mathcal{A}$ - a $*$-alg.:
$$|\omega(a^* \cdot b)|^2 \leq |\omega(a^* a)| \cdot |\omega(b^* b)| \quad \forall a, b \in \mathcal{A}.$$

- Since: $K(a,b) := \omega(a^* \cdot b)$ - a positive semidef. Hermitian form (check!)

<u>Remark</u> A very important form in the representation theory.

<u>Lemma 3.3.</u> Let $\mathcal{A}$ - unital C^*-alg., $a \in \mathcal{A}$. Then $\forall \lambda > 1$: $\lambda \|a\|^2 \cdot 1 - a^* a \geq 0$.

<u>Remark.</u> True even for $\lambda = 1$ (prove for an exercise) but we shall not use it.

<u>Proof.</u> Let $c := \sum_{n=0}^{\infty} \binom{1/2}{n} \Big(\underbrace{\frac{a^* a}{\lambda \|a\|^2}}_{\|\cdot\| < 1}\Big)^n$ - abs. converg. since: $\|\cdot\| < 1$ $\Bigg\} \Rightarrow$ $c^* = c$, $c^2 = 1 - \dfrac{a^* a}{\lambda \|a\|^2} \geq 0$. $\square$

<u>Proof of Theorem 3.1.</u> $0 \leq \omega(1^* 1) = \omega(1) = |\omega(1)| \leq \|\omega\| \|1\| = \|\omega\|$

$\Rightarrow \quad \omega(1) = |\omega(1)| \leq \|\omega\|$. Conversely, from Cor. 3.2:

$|\omega(a)|^2 = |\omega(1^* a)|^2 \leq |\omega(1^* \cdot 1)|\, |\omega(a^* a)| = \omega(1)\, \omega(a^* a)$. From Lemma 3.3:

$\forall \lambda > 0$: $0 \leq \omega(\lambda \|a\|^2 \cdot 1 - a^* a)$. $\Rightarrow \omega(a^* a) \leq \lambda \|a\|^2 \omega(1) \quad \forall \lambda > 1$.

$\Rightarrow$ true also for $\lambda = 1$: $\omega(a^* a) \leq \|a\|^2 \omega(1)$. $\Rightarrow |\omega(a)|^2 \leq \omega(1)^2 \|a\|^2$.

$\Rightarrow \|\omega\| \leq \omega(1)$. $\square$

Corollary 3.4. A-(unital) C^*alg, ω-(real) positive lin. funct.: $\omega(b^*a^*ab) \leq \|a\|^2 \omega(b^*b)$.
Hint: $b^*(\lambda\|a\|^2.1 - a^*a)b \geq 0$ for $\lambda > 1$ (why?). Then $\omega(-|-) \geq 0$. $\square$

C^*-operator algebras.

Hilbert space (Хилбертово просранство): elementary facts

Rudin, Real and Complex analysis, Ch. 4 § 1-2.
Рудин, Реален и комплексен анализ, Гл. 4 § 1-2.

1.) Pre-Hilbert space (предхилбертово просранство): this is a complex vector space V equipped with a Hermitian form $\langle u, v\rangle$, $u, v \in \mathbb{C}$ which is (strongly) positive definite:
positive semidef. & $\langle u, u\rangle \neq 0$ for $u \neq 0$.

2.) Then $\|u\| := \sqrt{\langle u, u\rangle}$ is a norm on V (by Cauchy-Schwarz ineq.)

3.) If V is a complete pre-Hilbert space $\overset{}{\underset{def}{\Longleftrightarrow}}$ V is Hilbert space.

4.) If V - pre-Hilb. sp., $W \underset{linear}{\subseteq} V$: $W^\perp := \{v \in V \mid \langle u, v\rangle = 0 \ \forall u \in W\}$

Then: $W_1 \subseteq W_2 \Rightarrow W_2^\perp \subseteq W_1^\perp$; $W \subseteq W^{\perp\perp}$ ← Galua correspondence
Both imply: $W^{\perp\perp\perp} = W^{\perp\perp}$ $\left(W^\perp \subseteq (W^\perp)^{\perp\perp} \ \& \ W^\perp \supseteq (W^{\perp\perp})^\perp\right)$.

5.) In a Hilb. space V, $W \underset{lin.}{\subseteq} V$: $W^{\perp\perp} = \overline{W}$ (the closure of W, затворена обвивка)

6.) In a Hilb. space V, $W \subseteq V$: $V = W \oplus W^\perp$ [Rudin, Th. 4.11].
linear and closed

7.) Riesz theorem (теорема на Рис): if $\varphi: V \to \mathbb{C}$ - bounded lin. funct. on a Hilb. sp. V, then $\exists!\, u \in V$ s.t. $\varphi(v) = \langle u, v\rangle$ - $\forall v \in V$.

Proof. Existence: $W := \operatorname{Ker}\varphi = \varphi^{-1}(0) \subseteq V$ - closed (inverse image of a closed set).
$\dim W^\perp = 1$: $u_1, u_2 \in W^\perp \Rightarrow \varphi(\underbrace{\varphi(u_1)u_2 - \varphi(u_2)u_1}_{\in W,\text{ but also } \in W^\perp}) = 0$; $W \cap W^\perp = 0$ $\Rightarrow (\cdots) = 0$

Let $u' \in W^\perp$, $u' \neq 0$. Set $u := \dfrac{\varphi(u')}{\langle u', u'\rangle} u'$; $V = W \oplus \mathbb{C}u'$.

$\Rightarrow v = v_0 + \gamma u'$. $\Rightarrow \varphi(v) = \underbrace{\varphi(v_0)}_{0} + \gamma\varphi(u')$; $\langle u, v\rangle = \underbrace{\langle u, v_0\rangle}_{0} + \gamma\langle u, u'\rangle = \gamma\varphi(u')$.

Uniqueness: $\langle u, v\rangle = \langle u', v\rangle$ $(\forall v)$ $\Rightarrow$ $\langle u - u', v\rangle = 0$ $(\forall v)$
$\Rightarrow u - u' = 0$. $\square$

8.) Let $a: V \to V$ bounded lin. operator in a Hilb. sp., Hermitian: $\langle a^*(u), v\rangle = \langle u, a(v)\rangle$

Then: $\|a\| \underset{\text{def}}{=} \sup_{\|v\|=1} \|a(v)\| = \sup_{\|v\|=1} |\langle v, a(v)\rangle|$

Proof. $C_a := \sup_{\|v\|=1} |\langle v, a(v)\rangle| \underset{\text{Cauchy-Schwarz}}{\leq} \sup_{\|v\|=1} \|a(v)\| = \|a\|$. Conversely

$u = a(v)$, $\lambda := \left(\frac{\|a(v)\|}{\|v\|}\right)^{1/2}$

$\|a(v)\|^2 = \langle a(v), a(v)\rangle$

$= \langle a(\lambda v), \lambda^{-1} u\rangle = \frac{1}{4}\left(\langle a(\lambda v + \lambda^{-1} u), \lambda v + \lambda^{-1} u\rangle - \langle a(\lambda v - \lambda^{-1} u), \lambda v - \lambda^{-1} u\rangle\right)$

$\leq \frac{1}{4} C_a \left(\|\lambda v + \lambda^{-1} u\|^2 + \|\lambda v - \lambda^{-1} u\|\right) = \frac{1}{2} C_a \left(\|\lambda v\|^2 + \|\lambda^{-1} u\|^2\right)$

$= \frac{1}{2} C_a \left(\lambda^2 \|v\|^2 + \lambda^{-2} \|a(v)\|^2\right) = C_a \|v\| \, \|a(v)\|$

$\Rightarrow \|a(v)\| \leq C_a \|v\| \quad (\forall v \in V). \Rightarrow \|a\| \leq C_a$. $\square$

9) Corollary: $\mathcal{B}(V) := \{a: V \to V \mid a\text{-bounded lin.}\}$ is a C^*-alg. with a^*: $\langle a^*(u), v\rangle = \langle u, a(v)\rangle$ (by Riesz thm.).

Proof. $\|a^* a\| = \sup_{\|v\|=1} \langle a^* a(v), v\rangle = \sup_{\|v\|=1} \langle a(v), a(v)\rangle = \sup_{\|v\|=1} \|a(v)\|^2 = \|a\|^2$

(to be continued)